XVI. *On the Roots of Equations. By* James Wood, *B.D. Fellow of St. John's College, Cambridge. Communicated by the Rev.* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

Read May 17, 1798.

The great improvements in algebra, which modern writers have made, are chiefly to be ascribed to Vieta's discovery, that "every equation may have as many roots as it has dimensions." This principle was at first considered as extending only to positive roots; and even when it was found that the number might, in some cases, be made up by negative values of the unknown quantity, these were rejected as useless. It could not, however, long escape the penetration of the early writers on this subject, that in many equations, neither positive nor negative values could be discovered, which, when substituted for the unknown quantity, would cause the whole to vanish, or answer the condition of the question. In such cases, the roots were said to be impossible, without much attention to their nature, or inquiry whether they admit of any algebraical representation or not. As far as the actual solution of equations was carried, *viz.* in cubics and biquadratics, the imaginary roots were found to be of this form, $a + \sqrt{-b^2}$; and subsequent writers, from this imperfect induction, concluded in general, that every equation has as many roots, of the form $a \pm \sqrt{\pm b^2}$, as it has dimensions. In the present state of the science, this proposition is of considerable importance, and its truth ought to be established on surer grounds. The various transformations of equations, the dimensions to which they rise in their reduction, and the circumstances which attend their actual solution, are most easily explained, and most clearly understood, by the help of this principle. Mr. EULER appears to have been the first writer who undertook to give a general proof of the proposition; but, whatever may be thought of his reasoning in other respects, as he carries it no further than to an equation of four dimensions, and it does not appear capable of being easily applied in other cases, it gives us no insight into the subject. Dr. WARING'S observations upon the proposition are extremely concise;* and, to common readers, it will still be a matter of doubt, whether a quantity of any description whatever will, when substituted for x in the expression $x^8 - px^7 + qx^6 - \ldots\ldots + w$, cause the whole to vanish.

In the investigation of the proof here offered, it became necessary to attend to the method of finding the common measure of two algebraical expressions; and to observe particularly, in what manner new values of the indeterminate quantities are introduced; and how they may again be rejected. It appears, that these values are necessary in the division; and, when they have been thus introduced, they enter every term of the *second* remainder, from which they may be discarded. This circumstance enables us, not only to determine the nature of the roots of every equation, but also affords us a direct and easy method of reducing any number of equations to one, and obtaining the final equation in its lowest terms.

* *Meditationes Alg.* p. 272.

PROP. I.

To find a common measure of the quantities $ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} +$ &c. and $Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} +$ &c.

In order to avoid fractions, multiply every term of the dividend by A^2, the square of the coefficient of the first term of the divisor, and the operation will be as follows:

$$\begin{array}{l|l} Ax^{n-1} + Bx^{n-2} & aA^2x^n + bA^2x^{n-1} + cA^2x^{n-2} + dA^2x^{n-3} + \text{\&c.} \;\; (aAx + bA - aB \\ + Cx^{n-3} + \text{\&c.} & aA^2x^n + aBAx^{n-1} + aCAx^{n-2} + aDAx^{n-3} + \text{\&c.} \\ \hline & * \; (bA^2 - aBA)x^{n-1} + (cA^2 - aCA)x^{n-2} + (dA^2 - aDA)x^{n-3} + \text{\&c.} \\ & \;\;\; (bA^2 - aBA)x^{n-1} + (bBA - aB^2)x^{n-2} + (bCA - aBC)x^{n-3} + \text{\&c.} \\ \hline (P) & * \;\; (cA^2 - bBA + aB^2 - aCA)\,x^{n-2} + \\ & (dA^2 - bCA + aBC - aDA)\,x^{n-3} + \text{\&c.} \end{array}$$

Let $(cA - bB)\,A + (B^2 - CA)\,a = \alpha$

$(dA - bC)\,A + (BC - DA)\,a = \beta$

$(eA - bD)\,A + (BD - EA)\,a = \gamma$ &c.

and the first remainder (P) is $\alpha x^{n-2} + \beta x^{n-3} + \gamma x^{n-4} +$ &c. proceed with this as a new divisor, and the next remainder (Q) will be $(\overline{C\alpha - B\beta}\,.\,\alpha + \overline{\beta^2 - \alpha\gamma}\,.\,A)\,x^{n-3} + (\overline{D\alpha - B\gamma}\,.\,\alpha + \overline{\beta\gamma - \alpha\delta}\,.\,A)\,x^{n-4} +$ &c.

Respecting this operation we may observe:

1. That were not every term of the first dividend multiplied by A^2, that quantity would be introduced by reducing the terms of the remainder (P) to a common denominator.

2. When $P = o$, $Ax^{n-1} + Bx^{n-2} + Cx^{n-3} +$ &c. is a divisor of $A^2\,(ax^n + bx^{n-1} + cx^{n-2} +$ &c.$)$; and therefore it is a divisor of $ax^n + bx^{n-1} + cx^{n-2} +$ &c. unless it be a divisor

of A^2, which is impossible; consequently no alteration is, in this case, made in the conclusion, by the introduction of A^2.

3. When P does not vanish, then every divisor of P is a divisor of $Ax^{n-1} + Bx^{n-2} + Cx^{n-3} +$ &c. and of A^2 $(ax^n + bx^{n-1} + cx^{n-2} +$ &c.); and therefore of $ax^n + bx^{n-1} + cx^{n-2} +$ &c. unless $A^2 = o$, in which case the remainder, P, becomes $aB(Bx^{n-2} + Cx^{n-3} +$ &c.), every divisor of which is a divisor of $Bx^{n-2} + Cx^{n-3} +$ &c. whether it be a divisor of $ax^n + bx^{n-1} + cx^{n-2} +$ &c. or not. That is, there are two values of the indeterminate quantity A, which, if retained, will produce erroneous conclusions.

4. A^2 enters every term of the second remainder (Q), and the two values, before introduced, may therefore be again rejected.

The coefficient of the first term of this remainder is $\overline{C\alpha - B\beta}.\alpha + \overline{\beta^2 - \alpha\gamma}.A$; and, by substituting for α, β and γ, their values, and retaining only those terms in which A is not found, and those in which it is only of one dimension, we have

$$C\alpha = -bBCA + aCB^2 - aC^2A$$

$$-B\beta = +bBCA - aCB^2 + aBDA$$

$$C\alpha - B\beta = -aC^2A + aBDA$$

$$\overline{C\alpha - B\beta}.\alpha = -a^2B^2C^2A + a^2B^3DA$$

$$\overline{\beta^2 - \alpha\gamma}.A = a^2B^2C^2A - a^2B^3DA;$$

therefore, those parts of $\overline{C\alpha - B\beta}.\alpha + \overline{\beta^2 - \alpha\gamma}.A$, in which A is of one dimension, and in which it is not found, vanish.

In the same manner it appears, that A^2 enters every other term of the remainder Q.

5. If the remainder $Q = o$, then, by the second observation, the introduction of α^2 in the division, produces no error in the conclusion; and, if Q do not vanish, α^2 will be found in every term of the third remainder, and may there be rejected; and so on. Thus we obtain the conclusion, without any unnecessary values of A, B, C, *&c.* or *a*, *b*, *c*, *&c.*

6. If the highest indices of x, in the original quantities, be equal, it will only be necessary to multiply the terms of the dividend by A, which may be rejected after the second division. If the difference of the highest indices of x be m, the terms of the dividend must be multiplied by A^{m+1}, the first quotient being carried to $m+1$ terms. This quantity, A^{m+1}, will enter every factor in each term of the second remainder.

7. If it be necessary to continue the division, let

$$\overline{C\alpha - B\beta}.\alpha + \overline{\beta^2 - \alpha\gamma}.A = mA^2$$
$$\overline{D\alpha - B\gamma}.\alpha + \overline{\beta\gamma - \alpha\delta}.A = nA^2$$
$$\overline{E\alpha - B\delta}.\alpha + \overline{\beta\delta - \alpha\varepsilon}.A = pA^2$$
&c.

and the third remainder is $(\overline{\gamma m - \beta n.m} + \overline{n^2 - mp.\alpha})\,x^{n-4} + (\overline{\delta m - \beta p.m} + \overline{np - mq.\alpha})x^{n-5} + (\overline{\varepsilon m - \beta q.m} + \overline{nq - mr.\alpha})\,x^{n-6} +$ *&c.* every term of which is divisible by α^2. The law of continuation is manifest.

PROP. II.

Two roots of an equation of $2m$ dimensions may be found by the solution of an equation of $m.\overline{2m-1}$ dimensions.

Let $x^{2m} + px^{2m-1} + qx^{2m-2} + rx^{2m-3} +$ *&c.* $= 0$; and, if possible, let v and z be so assumed that $v + z$, and $-v + z$, may be two roots of this equation; then,

$$\left.\begin{array}{lllll} x^{2m} & = v^{2m} \pm 2mzv^{2m-1} + 2m.\frac{2m-1}{2}.z^2v^{2m-2} & \pm 2m.\frac{2m-1}{2}.\frac{2m-2}{3}.z^3v^{2m-3} + \&c. \\ px^{2m-1} & = \pm pv^{2m-1} + \overline{2m-1}.pzv^{2m-2} & \pm \overline{2m-1}.\frac{2m-2}{2}.pz^2v^{2m-3} + \&c. \\ qx^{2m-2} & = \qquad qv^{2m-2} & \pm \overline{2m-2}.qzv^{2m-3} + \&c. \\ rx^{2m-3} & = & \pm rv^{2m-3} + \&c. \end{array}\right\} = 0$$

and consequently,

$$\left.\begin{array}{l} v^{2m} + 2m.\frac{2m-1}{2}.z^2 \\ + \overline{2m-1}.pz \\ + q \end{array}\right\} v^{2m-2} + \&c. + \left.\begin{array}{l} z^{2m} \\ + pz^{2m-1} \\ + qz^{2m-2} \\ + rz^{2m-3} \\ + \&c. \end{array}\right\} = 0$$

and also,

$$\left.\begin{array}{l} 2mz \\ + p \end{array}\right\} v^{2m-1} + \left.\begin{array}{l} 2m.\frac{2m-1}{2}.\frac{2m-2}{3}.z^3 \\ + \overline{2m-1}.\frac{2m-2}{2}.pz^2 \\ + \overline{2m-2}.qz \\ + r \end{array}\right\} v^{2m-2} + \&c. + \left.\begin{array}{l} 2mz^{2m-1} \\ + \overline{2m-1}.pz^{2m-2} \\ + \overline{2m-2}.qz^{2m-3} \\ + \overline{2m-3}.rz^{2m-4} \\ + \&c. \end{array}\right\} v = 0$$

Assume $y = v^2$; and let the coefficients of the terms of the former equation be 1, *b*, *c*, *d*, *&c.* and of the latter, A, B, C, D, *&c.* and the equations become

$$y^m + by^{m-1} + cy^{m-2} + dy^{m-3} + \&c. = 0$$
$$Ay^{m-1} + By^{m-2} + Cy^{m-3} + \&c. = 0.$$

These equations have a common measure of the form $y \pm Z$, where Z is expressed in terms of z and known quantities; and this common measure may be found, by dividing, as in Prop. 1. till y is exterminated, and making the last remainder equal to nothing.

Now, the first remainder is $(\overline{cA - bB}.A + B^2 - CA)\,y^{m-2} + (\overline{dA - bC}.A + BC - DA)\,y^{m-3} + (\overline{eA - bD}.A + BD - EA)\,y^{m-4} + \&c.$; or, by substitution, $\alpha y^{m-2} + \beta y^{m-3} + \gamma y^{m-4} + \&c.$

and, in α, z rises to 6 dimensions; in β, to 8 dimensions; and, in γ, to 10 dimensions, &c.

The second remainder is $(\overline{C\alpha - B\beta}\,.\,\alpha + \overline{\beta^2 - \alpha\gamma}\,.\,A)\,y^{m-3} + (\overline{D\alpha - B\gamma}\,.\,\alpha + \overline{\beta\gamma - \alpha\delta}\,.\,A)\,y^{m-4} +$ &c.; or, by substitution, $m\,A^2 y^{m-3} + n\,A^2 y^{m-4} +$ &c. and, dividing by A^2, the dimensions of z in m, are 15; in n, 17, &c. Let $\pi, \varkappa, \varrho, \sigma, \tau,$ &c. be the dimensions of z in the first term of the 1st, 2d, 3d, 4th, 5th, &c. remainders; then

$$\begin{aligned} \pi &= 6 \\ \varkappa &= 15 \\ \varrho &= 2\varkappa - \pi + 4 \\ \sigma &= 2\varrho - \varkappa + 4 \\ \tau &= 2\sigma - \varrho + 4 \\ &\text{\&c.} \end{aligned}$$

the increment of the $m - 1$ term of this series is $4m + 1$, and therefore the $m - 1$ term itself is $2m\,.\,\overline{m - 1} + m$, or $m\,.\,\overline{2m - 1}$. Now, in the $m - 1$ remainder, y does not appear, and, in that remainder, z rises to $m\,.\,\overline{2m - 1}$ dimensions; if then, this remainder be made equal to nothing, and a value of z determined, the last divisor, $y \mp Z$, where Z is some function of z, is known; and this is a common measure of the two equations $y^m + by^{m-1} + cy^{m-2} +$ &c. $= 0$, and $Ay^{m-1} + By^{m-2} + Cy^{m-3} +$ &c. $= 0$; consequently, $y \mp Z = 0$; and $y = \pm Z$; hence $\pm\sqrt{y}$, or $v, = \pm\sqrt{\pm Z}$; therefore, by the solution of an equation of $m\,.\,\overline{2m - 1}$ dimensions, two roots, $z \pm \sqrt{\pm Z}$, of the original equation, are discovered.

Cor. 1. Since two roots of the proposed equation are $z + v$, and $z - v$, $x^2 - 2zx + z^2 - v^2 = 0$ is a quadratic factor of that equation.

Cor. 2. In the same manner that the two equations

$y^m + by^{m-1} + cy^{m-2} +$ &c. $= 0$, and $Ay^{m-1} + By^{m-2} + Cy^{m-3} +$ &c. $= 0$, are reduced to one, may any two equations be reduced to one, and one of the unknown quantities exterminated; also, the conclusion will be obtained in the lowest terms.

PROP. III.

Every equation has as many roots, of the form $a \pm \sqrt{\pm b^2}$, as it has dimensions.

Case 1. Every equation of an odd number of dimensions has, at least, one possible root; and it may, therefore, be depressed to an equation of an even number of dimensions.

Case 2. If the equation be of $2m$ dimensions, and m be an odd number, then $m . \overline{2m - 1}$ is an odd number, and consequently z and v^2 (see Prop. II.) have possible values; therefore the proposed equation has a quadratic factor, $x^2 - 2zx + z^2 - v^2 = 0$, whose coefficients are possible; that is, it has two roots of the specified form; and it may be reduced two dimensions lower.

Case 3. If m be evenly odd, or $\frac{m}{2}$ an odd number, then the equation for determining z, has either two possible roots, or two of the form $a \pm b\sqrt{-1}$, (Case 2.); and v^2 will be of the form $c \pm d\sqrt{-1}$; hence, one value of the quadratic factor $x^2 - 2zx + z^2 - v^2 = 0$, will be of this form, $x^2 - \overline{2a + 2b\sqrt{-1}}.x + AB + CD\sqrt{-1} = 0$; and another of this form, $x^2 - \overline{2a - 2b\sqrt{-1}}.x + AB - CD\sqrt{-1} = 0$; consequently, $x^4 - 4ax^3 + (2AB + 4a^2 + 4b^2)x^2 - (4aAB + 4bCD)x + A^2B^2 + C^2D^2 = 0$, will be a factor of the proposed equation; and this biquadratic may be resolved into two quadratics, whose coefficients are possible, and whose roots are therefore of the form specified in the proposition.

In the same manner the proposition may be proved, when $\frac{m}{4}$, $\frac{m}{8}$, $\frac{m}{16}$ &c. is an odd number; and thus it appears that it is true in all equations.*

Cor. 1. If v^2, or y, be positive, the roots of the quadratic factor $x^2 - 2zx + z^2 - v^2 = 0$, and therefore two roots of the proposed equation are possible. If $y = 0$, two roots are equal; and if y be negative, two roots are impossible.

Cor. 2. If a possible value of z be determined, and substituted in b, c, d, &c. the original equation will have as many pairs of possible roots as there are changes of signs in the equation $y^m + by^{m-1} + cy^{m-2} + \&c. = 0$; and as many pairs of impossible roots as there are continuations of the same sign.

* See Dr. WARING's Med. Alg.